MODIFIED VARIATIONAL ITERATION METHOD FOR THE SOLUTIONS OF HESTON STOCHASTIC PARTIAL DIFFERENTIAL EQUATION USING THE MAMADU-NJOSEH ORTHOGONAL POLYNOMIALS

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The Variational Iteration Method (VIM) is a powerful tool in the field of numerical analysis in determining the numerical solutions of linear and non-linear differential equations. In this paper, a modified form of the VIM using the Mamadu-Njoseh polynomials (MNPs) as modifier and basis function was presented to approximate the Heston Stochastic Partial Differential Equation to its exact solution. Results obtained were compared with that of VIM available in literature and it showed that results obtained from the modified VIM converges faster than that of VIM.

Key words: Variational Iteration Method (VIM), orthogonal polynomials, Mamadu-Njoseh Polynomials (MNPs), Heston Stochastic Partial Differential Equation (HSPDE)

INTRODUCTION

He (1999) developed an iterative method for solving differential and integral equations. This method, the Variational Iteration Method (VIM), was devoid of the limitations of the Grid point techniques, the Spline solution, the Perturbation method and the Adomian Decomposition method (He et al., 2007). The method could treat linear and non-linear equations alike, without any unrealistic assumptions. Safari (2011) applied the VIM to derive the analytic solution space fractional diffusion equation, while Baghani et al. (2012) employed the method on the non-linear free vibration of conservative oscillator. Elsheikh and Elzaki (2016) applied a modified form of the VIM to solve a fourth order parabolic partial differential equation with variable coefficients, while Abassy (2011) applied a modified VIM to obtain the solution of some non-linear, nonhomogeneous differential equations. Biazar et al. (2015) compared the VIM against the ADM and HPM on the numerical solutions of the Heston partial differentia equation where they concluded that the method is much easier, more convenient, more stable and efficient than the other two iterative methods. Mamadu-Njoseh Polynomials (MNPs) was developed and employed on the numerical solutions of fifth order boundary value problems by Njoseh and Mamadu (2016). The MNPs, a $C^{[a,b]}$ orthogonal polynomial was also used to derive the numerical solutions of the Voltera equation, using Galerkin method (Mamadu and Njoseh, 2016). This paper is geared towards using the MNPs as a modifier and basis function to the VIM in arriving at the approximate solution of Heston stochastic partial differential equation via the orthogonal collocation method. The swift convergence of the MVIM is represented graphically.

Heston Stochastic Partial Differential Equation (HSPDE)

According to Alziary and Takac (2017), the Heston model is a typical stochastic volatility model of the form

$$\alpha(t, S(t), V(t)) = (a - bV(t))$$

and

$$(t, S(t)), V(t)) = \sigma\sqrt{V(t)}$$

While, Biazar et al. (2015) gives the Heston model as

$$\frac{dS(t)}{S(t)} = rdt + \sigma\sqrt{V}d\widetilde{W}_1(t)$$

$$\frac{dV(t)}{d(t)} = \big(a - bV(t)\big) + \sigma\sqrt{V}d\widetilde{W}_2(t) \quad (1)$$

where α is the option price, β is the price of the volatility risk, r is the interest rate, $S(t)$ is the asset price at time t, $V(t)$ is the volatility of the asset price at time t with $\sqrt{V}$ as the variance of the volatility, a is the long-run mean, b is the speed of the mean reversion, σ is the volatility of the variance process, while $d\widetilde{W}_1(t)$ and $d\widetilde{W}_2(t)$ are correlated Brownian motions under the risk-neutral measure with the correlation coefficient $\rho\epsilon(-1,1)$ such that

$$d\widetilde{W}_1(t)d\widetilde{W}_2(t) = \rho dt \quad (2)$$

The risk-neutral price of a call expiring at time $t \le T$ in the Heston stochastic volatility model is given as

$$c\big(t, S(t), V(t)\big) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^{+}], 0 \le t \le T \quad (3)$$

The equation

$$\frac{\partial c}{\partial t} + rs\frac{\partial c}{\partial s} + (a - bv)\frac{\partial c}{\partial v} + \frac{1}{2}s^2 v\frac{\partial^2 c}{\partial s^2} + \rho\sigma sv\frac{\partial^2 c}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c}{\partial v^2} - rc = 0 \quad (4)$$

is the Heston partial differential equation (PDE) for the fair values of European style options, forming a time dependant convection diffusion reaction equation with mixed spatial derivative terms.
The Heston PDE (4) has the initial and boundary conditions given as

$$C(S, v, t) = \max(0, s - K)$$
$$C(0, v, t) = 0 \quad (5)$$

where $k > 0$ is the given strike price.

He's Variational Iteration Method (VIM)
He (1999) developed a Variational Iteration Method (VIM) in solving linear and nonlinear equations alike without any unrealistic assumptions. He et al. (2007) concluded that the method converges faster to the exact solution by successive approximation. Unlike the Adomian Decomposition Method (ADM) developed by George Adomian between 1980 and 1988, the VIM does not require any form of polynomials to obtain its approximate solution. According to He et al. (2010), the basic concept of the VIM is to construct a correction functional for nonlinear systems. This is given as:

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t \lambda\Big(LU_n(\tau) + N\widetilde{U}_n(\tau) - g(\tau)\Big)d\tau \quad (6)$$

where λ is a general langrange multiplier, which can be identified optimally via the variational theory. $N\widetilde{U}_n(\tau)$ is considered as the restricted variations. If the language multiplier $\lambda = 1$, then Equation 6 will be given as

$$U_{n+1}(x,t) = U_n(x,t) - \int_0^t \big(LU_n(\tau) + NU_n(\tau) - g(\tau)\big)d\tau \quad (7)$$

$NU_n(\tau)$ is called the correction term and Equation 7 can be solved iteratively using $U_0(x)$ as the initial approximation with possible unknowns.

Mamadu-Njoseh polynomials (MNPs)
The MNPs are a set of orthogonal polynomials having an interval of $[-1, 1]$ and a weight function of $w(x) = (1 + x^2)$. It is given as:

$$\int_{-1}^{1} \varphi_m(x)\varphi_n(x)(1 + x^2)dx = 0 \quad (8)$$

with the first four MNPs given as:

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = x$$
$$\varphi_2(x) = \frac{1}{3}(5x^2 - 2)$$
$$\varphi_3(x) = \frac{1}{5}(14x^3 - 9x) \quad (9)$$

Modified VIM for Heston SPDE

The He's VIM was modified for the study using the MNPs as modifier and basis function via the orthogonal collocation method (OCM). Given the general formulation of the VIM in Equation 6, the correction function as related to the HSPDE is thus given as:

$$c_{n+1}(s,v,t) = c_n(s,v,t) + \int_0^t \lambda(\xi)\left[\frac{\partial c}{\partial \xi} - rs\frac{\partial c}{\partial s} + (a-bv)\frac{\partial c}{\partial v} + \frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2} + \rho\sigma sv\frac{\partial^2 c}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c}{\partial v^2} + rc\right]d\xi \quad (10)$$

The scheme for generating the initial approximation through the OCM with the Mamadu-Njoseh polynomials as basis function is described as follows:

Let the initial approximation be given as

$$c_{(0)} = \sum_{i=0}^{N} a_i\varphi_i(x), \quad (11)$$

where a_i are unknown constants to be determined and $\varphi_i(x)$ are the Mamadu-Njoseh Polynomials with interval of orthogonality $[-1,1]$. According to Biazar et al. (2015), the Heston SPDE has a generalized initial condition

$$c_0(x) = 2s^2t^2 \quad (12)$$

Incorporating Equation 11 and 12, the following is obtained

$$c_{(0)} = \sum_{i=0}^{N} a_i\varphi_i(x) = 2s^2t^2 \quad (13)$$

Solving Equation 13 at $N = 3$ (chosen arbitrarily) and substituting the $\varphi_i(x)$, $i = 0,1,2,3$, the following is obtained:

$$a_0 + a_1 s + a_2\left(\frac{5}{3}s^2 - \frac{2}{3}\right) + a_3\left(\frac{14}{5}s^3 - \frac{9}{5}s\right) = 2s^2t^2 \quad (14)$$

In Biazar et al. (2015), the value of t is defined within the range $0 \le t \le T$. Thus, collocating (14) at the zeroes of $\varphi_4(x)$,

$$s = 0.3676425560, -0.3676425560, 0.875671$$

and writing the resulting linear algebraic equations in the form

$$A\underline{X} = \underline{b} \quad (15)$$

where ,

$$A = \begin{bmatrix} 1 & 0.3676425560 & -0.4413982517 & -0.5226219309 \\ 1 & -0.3676425560 & -0.4413982517 & 0.5226219309 \\ 1 & 0.8756710201 & 0.6113328923 & 0.303892222 \\ 1 & -0.8756710201 & 0.6113328923 & -0.303892222 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 52.98313121 \\ 52.98313121 \\ 300.5854963 \\ 300.5854963 \end{bmatrix}$$

Solving Equation 15, using Gaussian elimination method, the following values were obtained for the constant $a_i's$

$$a_0 = 156.800000$$
$$a_1 = 0.000000$$
$$a_2 = 235.200000$$
$$a_3 = 0.000000$$

Thus, substituting the above in Equation 12, the following is obtained

$$c_{(0)} = 392.00000s^2 \quad (16)$$

Hence, the initial approximation for the modified VIM is given by Equation 16.

From Equation 10, it was seen that $\lambda(\xi)$ is the general langrange multiplier, which can be obtained optimally via the variational theory. If the following is set

$$\lambda(\xi) = -1 \quad (17)$$

and substituting Equation 17 into Equation 10 gives,

$$c_{n+1}(s,v,t) = c_n(s,v,t) - \int_0^t \left[\frac{\partial c}{\partial \xi} - rs\frac{\partial c}{\partial s} + (a-bv)\frac{\partial c}{\partial v} + \frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2} + \rho\sigma sv\frac{\partial^2 c}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c}{\partial v^2} + rc\right]d\xi \quad (18)$$

which is the MVIM with initial approximation $c_0(s, v, t) = 392.00000s^2$.

Evaluating Equation 18 with the aid of MAPLE 18 application software for $n \geq 0$ and with the following parametric values $a = 0.16, b = 0.055, \delta = 0.9, \rho = -0.5, T = 15, K = 100$ yields the following approximation as shown in Table 1. Hence the approximate solution of the HSPDE given by the MVIM as seen in Figures 1 and 2.

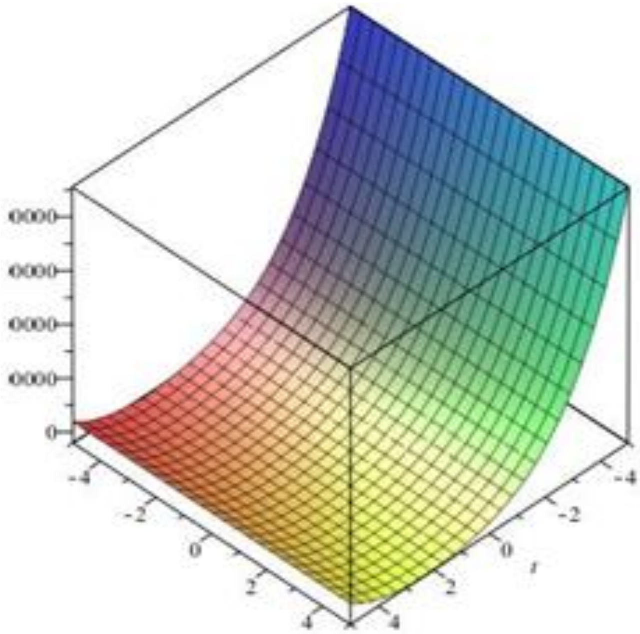

Figure 1. MVIM for HSPDE.

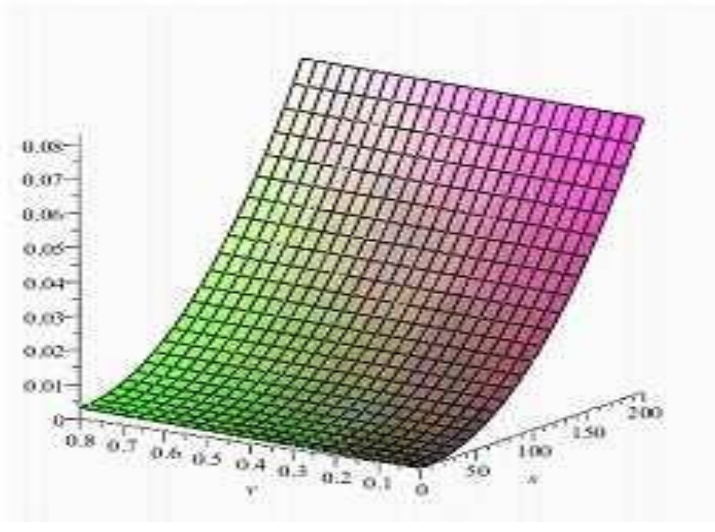

Figure 2. VIM for HSPDE.
Source: Biazer et al. (2015).

CONCLUSION

The MNPs being a new orthogonal polynomial was used as a modifier and basis function for the He's VIM via the OCM after which the MVIM was applied on the HSPDE. The compatibility of the MNPs and the VIM created a new iterative method which was able to approximate the Heston Stochastic Partial Differential Equation to its exact solution.

Furthermore, as comparison was made

between the results of the VIM (Biazer et al., 2015) and the MVIM, it was discovered that the MVIM matched the results of the VIM as can be seen in the Table 1 and Figures 1 and 2. Hence the MVIM can be used as a perfect substitute and a better approximating method of the HSPDE to the VIM, the MNPs playing a major role in the achievement of this feat. The study recommends the use of the MVIM for the approximation of the HSPDE to its exact solution.

Table 1. Aid of MAPLE 18 application software.

C(t, s, v)	VIM (Biazar et al., 2015)	MVIM
C(1, 10, 0.1)	209.5038332	38456.89887
C(2, 50, 0.2)	82474.33729	9.740327147 10^5
C(4, 70, 0.3)	1.147814970 E6	1.873262944 10^6
C(6, 90, 0.4)	1.950175983 E7	3.017060928 10^6
C(8, 120, 0.5)	7.79650567 E7	5.276105707 10^6
C(10, 150, 0.6)	2.317375113 E8	8.112385120 10^6
C(14, 200, 0.8)	2.576414314 E9	1.380834321 10^7

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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