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NUMERICAL SOLUTION OF NONLINEAR SCHRODINGER EQUATION USING COLLOCATION METHOD WITH CHEBYSHEV POLYNOMIALS

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Abstract

The nonlinear Schrodinger equation (NSE) is a partial differential equation (PDE) with numerous applications in quantum mechanics. Analytical methods for the solution of NSE are almost impossible due to their complexity. Thus, the need for a numerical scheme to seek the approximate solution of the NSE. Hence, this paper considered the numerical solution of the NSE via the spectral collocation method (SCM) with Chebyshev orthogonal polynomials of the first kind. The scheme was efficiently constructed to solve the NSE equation with the aid of MAPLE 18, and numerical evidence compared with Adomian Decomposition Method (ADM), Homotopy Analysis Transform Method (HATM) and Residual Power Series Method (RPSM) as available in the literature. The findings show that the SCM is an effective solver for NSE with rapid convergence to the exact solution as the parameter t varies.

Keywords: Schrodinger equation, Orthogonal polynomials, Chebyshev polynomials, collocation method, partial differential equation

Introduction

Differential equations found their relevance in almost every ramification of human endeavor, they form essential scientific tools in modeling physical and practical problems in the biological sciences, engineering, agricultural sciences, physical and social sciences (Adebiyi and Fatumo, 2006). The Schrodinger equation (SE)is a nonlinear partial differential equation that forms the basic building block in quantum physics. It has various applications in quantum mechanics and applied mathematics. The Schrodinger equation can be used to model several natural phenomena such as light propagation in fiber optics, pulses with dispersive effects whose shapes are preserve inside fiber, formation of monster waves on the surface of the ocean, propagation of waves in channels, electronic structure of atoms and molecules, dynamics of a Bose-Einstein condensate, among others (Ogundare, 2009; Mocz andSucci, 2016; Aksoy*et al*., 2013).

Most partial differential equations that model real-life situations may not necessarily have a closed form solution;instead, numerical techniques are utilized to approximate the solutions. Over the years numerical methods adopted for the Schrodinger equation have consistently evolved for the analysis of physical phenomena (Balaram, 2016; Balaram*et al.*, 2016). Some of these numerical techniques include,the B-Spline finite element methods, meshless method using collocation with radial basis function (Aksoy*et al*, 2013; Montegranario*et al*., 2016), Crank-Nicolson schemes, Runge-Kutta methods in time, finite difference and finite element method in space (Antoine *et al.*, 2011), Hopscotch method, Fourier Pseudo-spectral method and Split-step Fourier Transform (Neveen*et al*., 2021).

This paper aims to apply the spectral collocation method adopting Chebyshevpolynomials as trial functions to solve numerically the nonlinear Schrodinger equation shown below:

$$iU_t + U_{xx} + \beta|U|^2U = 0, \qquad (1.1)$$

subject to the initial condition

$$U(x, 0) = g(x),$$

where β is a positive constant term and $U(x, t)$ is complex valued function of two real variables x, t.The Chebyshev nodes which are the zeros of the Chebyshev polynomial are used for the polynomial interpolation.

Spectral Collocation Method for Nonlinear Schrodinger Equation

We now consider the nonlinear Schrodinger equation of the form

$$iu_t + u_{xx} + \beta|u|^2u = 0, \qquad (2.1)$$

subject to the initial condition

$$u(x, 0) = g(x), \qquad (2.2)$$

where β is a positive constant term and $u(x,t)$ is complex valued function of two real variables x, t.

The nonlinear Schrodinger equation below in equation (2.1) - (2.2) was solved by AL-Shareef et al. (2016) using Adomian decomposition method (ADM)

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad (2.3)$$

with initial condition

$$u(x, 0) = e^{ix}, \quad (2.4)$$

with exact solution as $u(x,t) = e^{i(x+t)}$

The nonlinear Schrodinger equation below in equation (2.5) - (2.6) was solved by Zeliha, (2019) using Residual Power Series Method (RPSM) and Homotopy Analysis Transform Method (HATM)

$$iu_t + u_{xx} - 2|u|^2u = 0, \quad (2.5)$$

with initial condition

$$u(x, 0) = e^{ix}, \quad (2.6)$$

with exact solution as $u(x,t) = e^{(x-3t)i}$.

In reference to the focus of this work, we let the approximate solution to (2.1) be defined as (Njoseh and Mamadu, 2017a)

$$u_n(x,t) = \sum_{r=0}^{N} a_r T_r(x) \cong u(x,t) \quad (2.7)$$

where $T_r(x), r \geq 0$ is the r^th^Chebyshev polynomial (Njoseh and Mamadu, 2016a, 2016b and 2016c);Mamadu and Njoseh, 2016; Mamadu*et al.*, 2021)of the first kind valid in the interval $-1 \leq x \leq 1$ and $a_r, r \geq 0$ are expansion coefficient (constants) which will be determined.

Substituting equating (2.7) into (2.5), we have,

$$i\frac{\partial}{\partial t}\left(\sum_{r=0}^{N} a_r T_r(x)\right) + \frac{\partial^2}{\partial x^2}\left(\sum_{r=0}^{N} a_r T_r(x)\right) + \beta\left|\left(\sum_{r=0}^{N} a_r T_r(x)\right)\right|^2\left(\sum_{r=0}^{N} a_r T_r(x)\right) = 0 \quad (2.8)$$

We can now collocate at the zeros of $T_r(x)$ in equation (2.8) and obtain a set of $(n+1)$ equations in $(n+1)$ unknown comprising of $a_r, r \geq 0$ (Njoseh and Mamadu, 2017b). A matrix solver, which in this case, is the Gaussian elimination method is employed to solve the resulting linear algebraic equations for a unique determination of the unknown coefficients in the approximate solution. Substituting the known coefficients into (2.7) generates the required approximate solution for the problem (2.1).

Numerical Examples

Nigerian Journal of Science and Environment 2024 Volume 22 (2) 60 – 71 ISSN: 3043 – 4440
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We consider the following test problems to test the accuracy and convergenceof the method as discussed in the previous section. MAPLE 18 will be adopted to carry out all numerical computations and simulation via the prescribed methodology discussed earlier for various values of N.

Test Problem 1. Consider the equation nonlinear Schrodinger equation (Al-Shareef*et al*., 2016)

$$iu_t + u_{xx} + 2|u|^2 u = 0, \quad (3.1)$$

with initial condition

$$u(x, 0) = e^{ix}, \quad (3.2)$$

with exact solution as $u(x,t) = e^{i(x+t)}$.

Test Problem 2. Consider the equation nonlinear Schrodinger equation(Zeliha, 2019)

$$iu_t + u_{xx} - 2|u|^2 u = 0, \quad (3.3)$$

with initial condition

$$u(x, 0) = e^{ix}, \quad (3.4)$$

with exact solution as $u(x,t) = e^{(x-3t)i}$.

Applying spectral collocation method for $N = 3$, results are presented in the tables below with the aid of MAPLE 18.

Computational Results for Test Problem 1:

Table 4.1 Comparison of Result between Exact and Approximate solution at t=0.03

X	Exact solution	Approximate Solution	Absolute Error
0	0.9995500337	0.9958096307	3.022781247E-8
0.1	0.9915618937	0.9911757226	2.988046106E-8
0.2	0.9736663950	0.9772739981	2.964126513E-8
0.3	9.94604234435	0.9541044574	2.971214900E-8
0.4	0.9089657497	0.9216671004	3.020388637E-8
0.5	0.8628070705	0.8799619271	3.103871867E-8
0.6	0.8080275083	0.8289889375	3.1909185726E-8
0.7	0.7451744023	0.7687481316	3.23114580710E-8
0.8	0.6748757601	0.6992395075	3.162558031E-8
0.9	0.5978339823	0.6204630710	2.923069677E-8
1.0	0.548188450	0.5324188163	2.477303906E-8

Nigerian Journal of Science and Environment 2024 Volume 22 (2) 60 – 71 ISSN: 3043 – 4440
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Table 4.2 Comparison of Result between Exact and Approximate solution at t=0.1

X	Exact solution	Approximate Solution	Absolute Error
0	0.9950041653	0.9958096307	9.983666588E-8
0.1	0.9800665778	0.9911757226	9.9535044449E-8
0.2	0.9553364891	0.9772739981	9.941424557E-8
0.3	9.9210609940	0.9541044574	9.961300485E-8
0.4	0.8775825619	0.9216671004	1.001794704E-7
0.5	0.8253356149	0.8799619271	1.01098695E-7
0.6	0.7648421873	0.8289889375	1.018655644E-7
0.7	0.6967067093	0.7687481316	1.02266467E-7
0.8	0.6216099683	0.6992395095	1.016090190E-7
0.9	0.5403023059	0.6204630710	9.915106224E-8
1.0	0.4535961214	0.5324188163	9.406942452E-8

Table 4.3 Comparison of Result between Exact and Approximate solution at t=0.5

X	Exact solution	Approximate Solution	Absolute Error
0	0.8775825619	0.9958096307	4.937878966E-7
0.1	0.8253356149	0.9911757226	4.935810063E-7
0.2	0.7648421873	0.9772739981	4.937013922E-7
0.3	0.6967067093	0.9541044574	4.942287738E-7
0.4	0.6216099683	0.9216671004	4.951359945E-7
0.5	0.5403023059	0.8799619271	4.962574089E-7
0.6	0.4535961214	0.8289889375	4.972672907E-7
0.7	0.3623577545	0.7687481316	4.976701428E-7
0.8	0.2674988286	0.6992395095	4.968028959E-7
0.9	0.1699671429	0.6204630710	4.938472493E-7
1.0	0.07073720167	0.5324188163	4.878494957E-7

Table 4.4 Comparison of result between SCM and ADM at t=0.03, 0.1 and 0.5

X	SCM Absolute Error at t=0.03	ADM Absolute Error at t=0.03	SCM Absolute Error at t=0.1	ADM Absolute Error at t=0.1	SCM Absolute Error at t=0.5	ADM Absolute Error at t=0.5
0	3.022781247E-8	3.370059347E-8	9.983666588E-8	4.1661332570E-6	4.937878966E-7	2.5955039E-3
0.1	2.988046106E-8	3.370059347E-8	9.9535044449E-8	4.1660562770E-6	4.935810063E-7	2.5955040E-3
0.2	2.964126513E-8	3.381153649E-8	9.941424557E-8	4.1661206330E-6	4.937013922E-7	2.5955039E-3
0.3	2.971214900E-8	3.374685170E-8	9.961300485E-8	4.1660544070E-6	4.942287738E-7	2.5955038E-3
0.4	3.020388637E-8	3.377217790E-8	1.001794704E-7	4.1660937870E-6	4.951359945E-7	2.5955040E-3
0.5	3.103871867E-8	3.379127106E-8	1.01098695E-7	4.1661050360E-6	4.962574089E-7	2.5955038E-3
0.6	3.1909185726E-8	3.370356064E-8	1.018655644E-7	4.1660713930E-6	4.972672907E-7	2.5955040E-3
0.7	3.2311458071E-8	3.375440712E-8	1.02266467E-7	4.1660511770E-6	4.976701428E-7	2.5955038E-3
0.8	3.162558031E-8	3.373499667E-8	1.016090190E-7	4.1662206490E-6	4.968028959E-7	2.5955036E-3
0.9	2.923069677E-8	3.376684765E-8	9.915106224E-8	4.1661306230E-6	4.938472493E-7	2.5955037E-3
1.0	2.477303906E-8	3.370830758E-8	9.406942452E-8	4.1660782630E-6	4.878494957E-7	2.5955036E-3

Computational Results for Test Problem 2:

Table 4.5 Comparison of Result between Exact and Approximate solution at t=0.01

X	Exact solution	Approximate Solution	Absolute Error
0.01	0.999800067	0.5324651554	4.674419845E-7
0.02	0.999950004	0.5326041726	4.674524177E-7
0.03	1.000000000	0.5328358680	4.681245000E-7
0.04	0.999500004	0.5331602416	4.694552837E-7

0.05	0.9998000067	0.5335772933	4.714390003E-7

Table 4.6 Comparison of Result between Exact and Approximate solution at t=0.02

X	Exact solution	Approximate Solution	Absolute Error
0.01	0.9987502604	0.5324651554	4.679966045E-7
0.02	0.9992001067	0.5326041726	4.670246957E-7
0.03	0.9995500337	0.5328358680	4.667141664E-7
0.04	0.9998000067	0.5331602416	4.670663392E-7
0.05	0.9999500004	0.5335772933	4.680796352E-7

Table 4.7 Comparison of Result between Exact and Approximate solution at t=0.03

X	Exact solution	Approximate Solution	Absolute Error
0.01	0.9968017063	0.5324651554	4.695718276E-7
0.02	0.9975510003	0.5326041726	4.676234136E-7
0.03	0.9982005399	0.5328358680	4.663302456E-7
0.04	0.9987502604	0.5331602416	4.656978953E-7
0.05	0.9992001047	0.5335772933	4.657290809E-7

Table 4.8 Comparison of Result between Exact and Approximate solution at t=0.04

X	Exact solution	Approximate Solution	Absolute Error
0.01	0.9939560980	0.5324651554	4.721560316E-7
0.02	0.9950041653	0.5326041726	4.692441059E-7
0.03	0.9959527330	0.5328358680	4.669756142E-7
0.04	0.9968017063	0.5331602416	4.653601891E-7
0.05	0.9975510003	0.5335772933	4.644047823E-7

Table 4.9 Comparison of Result between Exact and Approximate solution at t=0.05

X	Exact solution	Approximate Solution	Absolute Error
0.01	0.9902159962	0.5324651554	4.757304723E-7
0.02	0.9915618937	0.5326041726	4.718747947E-7
0.03	0.9928086359	0.5328358680	4.686454418E-7
0.04	0.9939560980	0.5331602416	4.660557650E-7
0.05	0.9950041653	0.5335772933	4.641167187E-7

Table 4.10: Comparison of Results between SCM and HATM t=0.01, 0.02 and 0.03

X	SCM Absolute Error at t=0.01	HAMT Absolute Error at t=0.01	SCM Absolute Error at t=0.02	HAMT Absolute Error at t=0.02	SCM Absolute Error at t=0.03	HAMT Absolute Error at t=0.03
0.01	4.674419845E-7	2.28657E-1	4.679966045E-7	4.97242E-2	4.695718276E-7	7.52547E-1
0.02	4.674524177E-7	2.28623E-1	4.670246957E-7	4.9884E-1	4.676234136E-7	7.59856E-1
0.03	4.681245000E-7	2.28304E-1	4.667141664E-7	4.99449E-2	4.663302456E-7	7.65568E-1
0.04	4.694552837E-7	2.27701E-1	4.670663392E-7	4.99065E-2	4.656978953E-7	7.69625E-1
0.05	4.714390003E-7	2.26822E-1	4.680796352E-7	4.97691E-2	4.657290809E-7	7.71988E-1

Table 4.11: Comparison of Results between SCM and HATM t=0.04 and 0.05

X	SCM Absolute Error at t=0.04	HAMT Absolute Error at t=0.04	SCM Absolute Error at t=0.05	HAMT Absolute Error at t=0.05
0.01	4.721560316E-7	9.11458E-1	4.757304723E-7	9.05243E-2
0.02	4.692441059E-7	9.29106E-1	4.718747947E-7	9.25587E-2
0.03	4.669756142E-7	9.45461E-1	4.686454418E-7	9.46623E-2
0.04	4.653601891E-7	9.6036E-1	4.660557650E-7	9.68162E-2
0.05	4.644047823E-7	9.73656E-1	4.641167187E-7	9.90013E-2

Nigerian Journal of Science and Environment 2024 Volume 22 (2) 60 – 71 ISSN: 3043 – 4440
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Table 4.12: Comparison of Results between SCM and RPSM t=0.01, 0.02 and 0.03

X	SCM Absolute Error at t=0.01	RPSM Absolute Error at t=0.01	SCM Absolute Error at t=0.02	RPSM Absolute Error at t=0.02	SCM Absolute Error at t=0.03	RPSM Absolute Error at t=0.03
0.01	4.674419845E-7	1.01148E-6	4.679966045E-7	7.68811E-3	4.695718276E-7	2.4201E-4
0.02	4.674524177E-7	3.02372E-6	4.670246957E-7	6.45381E-5	4.676234136E-7	2.45929E-4
0.03	4.681245000E-7	5.00575E-6	4.667141664E-7	1.28354E-4	4.663302456E-7	7.31412E-4
0.04	4.694552837E-7	6.93777E-6	4.670663392E-7	1.90888E-4	4.656978953E-7	1.20959E-3
0.05	4.714390003E-7	8.80046E-6	4.680796352E-7	2.51515E-4	4.657290809E-7	1.67567E-3

Table 4.13: Comparison of Results between SCM and RPSM t=0.04 and 0.05

X	SCM Absolute Error at t=0.04	RPSM Absolute Error at t=0.04	SCM Absolute Error at t=0.05	RPSM Absolute Error at t=0.05
0.01	4.721560316E-7	2.02148E-3	4.757304723E-7	9.13486E-3
0.02	4.692441059E-7	1.94821E-3	4.718747947E-7	2.97814E-3
0.03	4.669756142E-7	2.06025E-3	4.686454418E-7	3.20833E-3
0.04	4.653601891E-7	4.08043E-3	4.660557650E-7	9.36275E-3
0.05	4.644047823E-7	6.05985E-3	4.641167187E-7	1.54236E-2

Nigerian Journal of Science and Environment 2024 Volume 22 (2) 60 – 71 ISSN: 3043 – 4440
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Discussion

We have implemented the spectral collocation method (SCM) for the numerical solutions of nonlinear Schrodinger equation. Basically, Chebyshev polynomials were employed as trial functions in the approximation of the analytic solution of these problems. SCM was solved along the analytic solution for comparison purposes. In the Test Problem 1 and 2, the SCM was found to converge rapidly to the exact solution in as much as the parameter t varies as shown in the Tables 4.1 to 4.3 and 4.5 to 4.9. For instance, maximum errors of order10^{-8} at all values of t were obtained for Test Problem 1, but the smaller the value of t, the better the rate of convergence. This suggests that convergence is time dependent. However, for Test Problem 2, maximum errors of order 10^{-7}at all values of t were recorded, which also shows rapid convergence at all points, but convergence result is not time dependent. Solutions to both test problems are said to be stable.

Also, for Test Problem 1, SCM was compared to the results obtained from ADM as computed by AL-Shareef et al. (2016). SCM recorded maximum errors of order $10^{-8}, 10^{-8}$ and 10^{-7} at t=0.03, 0.1 and 0.5 respectively why ADM recorded maximum errors of order $10^{-8}, 10^{-6}$ and 10^{-3} at t=0.03, 0.1 and 0.5 respectively. This shows that SCM have higher convergence rate than ADM.

Furthermore, for Test Problem 2, SCM was compared to the results obtained through RPSM and HATM by Zeliha (2019). RPSM achieved maximum errors of order $10^{-6}, 10^{-5}, 10^{-4}$ *and* 10^{-3} at t=0.01, 0.02, 0.03 and 0.05 and SCM recorded maximum error order of 10^{-6} at every value of t. Both schemes have the same convergence rate at $t=$0.01, but error increases as the value of t increases for RPSM, which implies that the lower the value of t, the better the convergence, but SCM maintain the same convergence rate at all points. However, for HATM, maximum error order of 10^{-2} at

$t = 0.02$ and 0.05 were reported as compared to SCM with maximum error order of 10^{-6} .

Conclusion

We have considered the numerical solution of nonlinear Schrodinger equation using SCM. So far, it has been well established that the method is an effective solver for nonlinear Schrodinger equations and is highly accurate. It is also evident that the method offers several advantages which include, among others; cost-effectiveness as no extra interpolation is required in order to achieve several outputs, ease of implementation, easy to program and excellent rate of convergence.

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