MAMADU-NJOSEH SPECTRAL COLLOCATION METHOD FOR FRACTIONAL KLEIN-GORDON EQUATION

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Abstract

This study considered the spectral collocation method for solving the fractional Klein-Gordon Equation (FKGE) (Caputo-sense) with the aid of Mamadu-Njoseh polynomials. The main characteristic is to convert the given problem into a system of algebraic equations which can be solved easily with any of the usual methods. To show the accuracy and efficiency of the method, a benchmark problem is implemented with different values of alpha at different time t and the results obtained were compared with that obtained from the Variational Iteration method (VIM) existing in the literature. The results of numerical tests confirm that the spectral collocation method is superior to the Variational iteration method and is highly accurate. All computational frameworks of this research were implemented with the aid of MAPLE 18.

Keywords: Fractional derivative, Collocation Method (SCM), Mamadu-Njoseh Polynomials (MNPs), Algebraic Equations.

1. Introduction

In order to understand fractional physical phenomena, there is a great need to identify the solutions of fractional differential models. Obtaining an analytical solution to the fractional nonlinear differential equation is a difficult task, so the use of a numerical method is required to seek the solution. Many researchers have proposed several ways of solving different fractional partial differential equations. Over the years, Fractional Partial Differential Equations (FPDE) have been widely used in the interpretation and modeling of many realisms matters that appear in applied mathematics and physics including fluid mechanics, electrical circuits, diffusion, damping laws, relaxation processes, mathematical biology (Kilbas *et al.,* 2006). Unlike the Integer-order derivatives, the Fractional derivatives provide more accurate models of real-world problems. In recent years, several methods have been used to solve the fractional Klein-Gordon equation. Hariharan (2013) solved the fractional Klein-Gordon equation using the Wavelet method. Hosseini et *al.,* (2017) modified the Kudryashov method for solving the conformable time-fractional Klein-Gordon equations with quadratic and cubic nonlinearities. Yang *et al.* (2019) solved the nonlinear Time Fractional Klein-Gordon equation using the spectral collocation method (SCM). Amin, *et al.,* (2020) carried out research to seek the numerical treatment of the time fractional Klein-Gordon equation using the Redefined Extended Cubic B-Spline Functions. Eman, *et al.*, (2020) applied the fractional reduced differential transform Method (FRDTM) for solving the nonlinear fractional Klein-Gordon equation (FKGE). Xiangmei, *et al.*, (2020) applied the transform-based localized RBF method and quadrature to seek the numerical solution of the linear time-fractional Klein-Gordon equation.

Hussaini *et al.,* (1989) considered the theory and application of spectral collocation methods to fluid dynamics. They also described the fundamentals and summarize results pertaining to spectral approximations of functions. Khater et al. (2008) applied the spectral collocation method based on differentiated Chebyshev polynomials to solve Burgers'-type equations. According to the study, the method offers better accuracy in comparison with other previous methods. Zarebnia and Jalili (2013) employed the spectral collocation method with the aid of Chebyshev polynomial of first kind to seek the solution of a class of nonlinear partial differential equations.

Bhrawy and Zaky (2016) proposed the shifted fractional order Jacobi orthogonal functions (SFJFs) based on the definition of the classical Jacobi polynomials. Uwaheren and Taiwo (2016) employed the orthogonal polynomials as basis functions to solve fractional order integrodifferential equations. For instance, Njoseh and Mamadu (2016a) applied these polynomials for the solutions of Nonlinear Volterra integrodifferential equations via the Modified Variational Homotopy perturbation method. Njoseh and Mamadu (2016b) used their polynomial as trial functions for the solution of fifth-order boundary value problems via the power series approximation method. They constructed orthogonal polynomials by assuming a quadratic weight function that is positive in the interval [-1, 1]. Zayed et al. (2020) introduced the shifted Legendre-type matrix polynomials of arbitrary fractional orders and their various applications utilizing Rodrigues matrix formulas. In this study, they used the fractional order of Rodrigues formula in other to provide further investigation on such Legendre polynomials from a different point of view. These orthogonal polynomials are implemented through an appropriate numerical scheme as basis function. Recently, Njoseh *et al*., (2020) used these polynomials to obtain a numerical approximation of the SEIR Epidemic Model with the aid of the Variational iteration orthogonal collocation method. In a similar manner, Mamadu and Tsetimi (2020) proposed a new approach to singular initial value problems using Perturbation by decomposition. Mamadu (2020) also used these polynomials as basis functions to seek the numerical solution to the Black-Scholes model.

The real question is: are the Mamadu-Njoseh polynomials efficient just like others in the literature for solving problems in differential equations? Due to this, a comparative study was carried out by Njoseh and Mamadu (2017) in other to investigate the efficiency in terms of convergence between Mamdu-

Njoseh and Chebyshev of first kind polynomials. Both polynomials were applied as basis functions for the numerical solutions of tenth order boundary value problem via the power series approximation method. It was observed that both polynomials exhibited similar rate of convergence. Also, it was observed that in some nonlinear problems, the Mamadu-Njoseh polynomials are superior to the Chebyshev of first-kind polynomials. Hence this study approximate solution of the fractional Klein-Gordon equation (FKGE) using Mamadu-Njoseh polynomials (MNPs). Hence, we intend to solve FKGE using MNPs and compared the results obtained with other polynomials and with existing similar methods to be able to analyze its convergence.

2. Fractional Calculus

Fractional calculus has emerged as a model for a broad range of non-classical phenomena in the applied sciences and engineering (Blumen *et al*., 1989; Bouchaud *et al*., 1990; Baecumer *et al*., 2001; Barkai *et al*., 2000). Along with the expansion of numerous and even unexpected recent applications of the operators of the classical fractional calculus, the generalized fractional calculus is another powerful tool stimulating the development of this field (Kilbas *et al*., 2006; Kiryakova, 2008).

2.1 The Caputo Fractional derivative

The Caputo fractional derivative operator is defined by

$$ {}_{u_0}^{C}D_u^{\alpha} f(u) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{u_0}^{u} \frac{f^n(u')}{(u-u')^{\alpha+1-n}} du, & (n-1) < \alpha \leq n, \\ \left(\frac{d}{dx}\right)^{n-1} f(u) \ \text{ if } & \alpha + 1 = n \end{cases} \tag{1} $$

Under natural condition on the function f, for $n \to \infty$ the Caputo derivative becomes a conventional n-th derivative of the function f.

Properties of the Caputo Fractional Derivative

Some properties of Caputo fractional derivative that will use in this research are stated as follows (Mamadu *et al*., 2021)

1. $D^{\alpha}C = 0, C$ is a constant (2)
2. $D^{\alpha}x^{n} = \begin{cases} 0, & n \in \mathbb{N}, \ n \geq \alpha, \\ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{\beta-\alpha}, if & n \in \mathbb{N} \text{ and } \ n < \alpha, \end{cases}$ (3)

   where $[\alpha] \geq \alpha$ and $N = \{0,1,2, \dots\}$.
3. $D^{\alpha}\left(\lambda f_1(x) + \mu f_2(x)\right) = \lambda D^{\alpha} f_1(x) + \mu D^{\alpha} f_2(x)$

3. Mamadu-Njoseh Spectral Collocation Method

Let $\omega^{1,1}(x) = (1-x)^1(1+x)^1 = (1-x^2)$ be a weight function of the Mamadu-Njoseh polynomials. The set of Mamadu-Njoseh polynomials $\left\{\varphi_n^{1,1}(x)\right\}_{n=0}^{\infty}$ forms a complete $L^2_{\omega^{1,1}}(-1,1)$ orthogonal system, where $L^2_{\omega^{1,1}}(-1,1)$ is a weighted space which is defined by

$$L^2_{\omega^{1,1}}(-1,1) = \{u: u \ is \ measurable \ and \ \|u\|_{\omega^{1,1}} < \infty\},$$

which is equipped with the norm

$$\|u\|_{\omega^{1,1}} = \left(\int_{-1}^{1} |u(x)^2| \, \omega^{1,1}(x) dx\right)^{\frac{1}{2}} \quad (4)$$

and inner product

$$\langle u, v\rangle_{\omega^{1,1}} = \int_{-1}^{1} u(x)v(x) \, \omega^{1,1}(x) dx \quad \forall \, u, v \in L^2_{\omega^{1,1}}(-1,1) \quad (5)$$

For a given $N \geq 0$, we denote by $\{\theta_k\}_{k=0}^{N}$ the Mamadu-Njoseh points, and by $\{\omega_k\}_{k=0}^{N}$ the corresponding Mamadu-Njoseh weights (i.e., $\{\omega^{1,1}\}_{k=0}^{N}$). Then the Mamadu-Njoseh-Gauss integration formula is

$$\int_{-1}^{1} f(x) \, \omega^{1,1}(x) dx \approx \sum_{k=0}^{N} f(\theta_k)\omega_k, \quad (6)$$

where $\omega_k = \omega^{1,1}(x_k)$. In similar way, we denote the Mamadu-Njoseh- Gauss points by $\left\{\tilde{\theta}_k\right\}_{k=0}^{N}$ and corresponding Mamadu-Njoseh weights by $\left\{\omega_k^{1,1}\right\}_{k=0}^{N}$. Then the Mamadu-Njoseh Gauss integration formula is given as

$$\int_{-1}^{1} f(x)\,\omega^{1,1}(x)dx \approx \sum_{k=0}^{N} f(\tilde{\theta}_k)\omega_k^{1,1},$$

where $\omega_k = \omega_k^{1,1}(x_k)$.

For any given positive integer N, the collocation points is denoted by $\{x_i^{1,1}\}_{k=0}^{N}$., which is the set of $(N+1)$Mamadu-Njoseh- Gauss points and it is corresponding to the weight $\omega_k^{1,1}(x)$.

Let P_N denote the space of all polynomials of degrees not above N for any $v \in C[-1,1]$, then the Mamadu-Njoseh interpolating polynomial $I_N^{1,1}v \in P_N$ satisfies

$$I_N^{1,1}v\left(x_i^{1,1}\right) = v\left(x_i^{1,1}\right), \qquad 0 \le i \le N$$

The Mamadu-Njoseh Interpolation polynomial can be written in the form

$$I_N^{1,1}v(x) = \sum_{i=0}^{N} v\left(x_i^{1,1}\right)F_i(x), \quad 0 \le i \le N$$

where $F_i(x)$ is the Mamadu-Njoseh interpolation basis function associated with $\{x_i^{1,1}\}_{i=0}^{N}$.

For the possible of applying the theory of orthogonal polynomials, we use change of variables to transfer interval $[0,t]$ to a fixed interval $I =: [-1,1]$,

$$\varphi(x,t) = \varphi(\theta), \qquad \theta = ax + \frac{bt^{1-\alpha}}{\Gamma(\alpha)}$$

3.1 SCM to FKGE

The Fractional Klein-Gordon's equation is of the form:

$$D_t^{\alpha}[u(x,t)] + aD_{xx}^{2}[u(x,t)] + bu(x,t) + cG\big(u(x,t)\big) = F(x,t), \quad 0 \le x \le L, t_o \le t \le T \quad (7)$$

with initial conditions

$$u(x,t_o) = \mu_1(x), \quad u_t(x,t_o) = \mu_2(x) \qquad (8)$$

$$u(0,t) = \mu_3(x), \qquad u(L,t) = \mu_4(x) \qquad (9)$$

where D_t^α represent the Caputo fractional time derivative operation of $u(x,t)$, a and b are constants, $u(x,t)$ denotes the displacement of the wave at (x,t), $\alpha\epsilon(1,2]$ is a fractional order of the time derivative, G is a nonlinear function, F is the source term, a, b and c are real numbers.

Let an approximation solution be given as:

$$u(x,t) \coloneqq \sum_{i=0}^{N} a_i\varphi_i \qquad (10)$$

Let $N = 3$, where a_i are unknowns to be determined and φ_i a Mamadu-Njoseh polynomial. Now substituting the value of $N = 3$ and expanding we have;

$$u(x,t) \coloneqq a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 \qquad (11)$$

Substituting the approximate solution into equation (7), we have

$$D_t^\alpha[\sum_{i=0}^{3} a_i\varphi_i] + aD_{xx}^2[\sum_{i=0}^{3} a_i\varphi_i] + b\sum_{i=0}^{3} a_i\varphi_i + cG(\sum_{i=0}^{3} a_i\varphi_i) = F(x,t) \qquad (12)$$

Expanding the approximate solution we obtain,

$$D_t^\alpha[a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3] + aD_{xx}^2[a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3]$$

$$+b[a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3] + cG(a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3) = F(x,t) \qquad (13)$$

Substituting Mamadu-Njoseh polynomial into (13) for $\varphi_i = 1,2,3$, we have

$$D_t^\alpha\left[a_0 + a_1 t + a_2\left(\frac{5}{3}t^2 - \frac{2}{3}\right) + a_3\left(\frac{14}{5}t^3 - \frac{9}{5}t\right)\right]$$

$$+aD_{xx}^2\left[a_0 + a_1 x + a_2\left(\frac{5}{3}x^2 - \frac{2}{3}\right) + a_3\left(\frac{14}{5}x^3 - \frac{9}{5}x\right)\right]$$

$$+b\left[a_0(1) + a_1(x) + a_2\left(\frac{5}{3}x^2 - \frac{2}{3}\right) + a_3\left(\frac{14}{5}x^3 - \frac{9}{5}x\right)\right]$$

$$+cG\left(a_0 + a_1 x + a_2\left(\frac{5}{3}x^2 - \frac{2}{3}\right) + a_3\left(\frac{14}{5}x^3 - \frac{9}{5}x\right)\right) = F(x,t) \qquad (14)$$

Simplifying and expanding equation (14) we have

$$D_t^\alpha\left[\frac{14}{5}a_3t^3 + \frac{5}{3}a_2t^2 + \left(a_1 - \frac{9}{5}a_3\right)t + a_0 - \frac{2}{3}a_2\right]$$

$$+aD_{xx}^{2}\left[\frac{14}{5}a_3x^3+\frac{5}{3}a_2x^2+\left(a_1-\frac{9}{5}a_3\right)x+a_0-\frac{2}{3}a_2\right]$$

$$+b\left[\frac{14}{5}a_3x^3+\frac{5}{3}a_2x^2+\left(a_1-\frac{9}{5}a_3\right)x+a_0-\frac{2}{3}a_2\right]$$

$$+cG\left(\frac{14}{5}a_3x^3+\frac{5}{3}a_2x^2+\left(a_1-\frac{9}{5}a_3\right)x+a_0-\frac{2}{3}a_2\right)=F(x,t) \quad (15)$$

But

$$D_t^{\alpha}(K)=0 \quad (16)$$

Where K is a constant and

$$D_t^{\alpha}(Kx^n)=K\left[\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}\right] \quad (17)$$

Now implementing the two properties of Caputo-fractional derivatives in (16) and (17) on (15), we have

$$D_t^{\alpha}\left(\frac{14}{5}a_3t^3\right)=\frac{14}{5}a_3\left[\frac{\Gamma(4)}{\Gamma(4-\alpha)}\right]=e_1$$

$$D_t^{\alpha}\left(\frac{5}{3}a_2t^2\right)=\frac{5}{3}a_2\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]=e_2$$

$$D_t^{\alpha}\left(\left(a_1-\frac{9}{5}a_3\right)t\right)=\left(a_1-\frac{9}{5}a_3\right)\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]=e_3$$

The result for D_{xx}^2 is

$$\left[\frac{84}{5}a_3x+\frac{10}{3}a_2\right]$$

Now substituting the fractional derivatives obtained into equation (15) we get a residual equation which shall collocate.

$$R:=\left[\frac{14}{5}a_3\left[\frac{\Gamma(4)}{\Gamma(4-\alpha)}\right]+\frac{5}{3}a_2\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]+\left(a_1-\frac{9}{5}a_3\right)\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}\right]\right]+\left[\frac{84}{5}a_3x+\frac{10}{3}a_2\right]$$

$$+a\left[\frac{14}{5}a_3x^3+\frac{5}{3}a_2x^2+\left(a_1-\frac{9}{5}a_3\right)x+a_0-\frac{2}{3}a_2\right]$$

$$+bG\left(\frac{14}{5}a_3x^3+\frac{5}{3}a_2x^2+\left(a_1-\frac{9}{5}a_3\right)x+a_0-\frac{2}{3}a_2\right)-F(x,t)=0 \quad (18)$$

Now collocating at N=4; we have

$$\Phi = [0.3676425560, -0.3676425560, 0.8756710201, -0.8756710201]$$

Solving $a_i's$ and substituting the values into the residual equations we get (n+1) equations which are solved by Gauss- elimination to arrive at the required approximate solution to the Fractional Klein-Gordon's equations.

4. Convergence Analysis of the Spectral Collocation Method

We consider the convergence analysis of Spectral Collocation method (SPM) as applied to (7).

Let $H=(a,b)\times[0,\tau]$, where H is Hilbert space. Also, let

$$D_t^{\alpha}y_{n+1}(x,t): H\to R,\ \ n\geq 0 \quad (19)$$

with

$$\int D_t^{\alpha}y_{n+1}(x,t)dtd\phi<+\infty, n\geq 0 \quad (20)$$

Theorem 4.1

Define $D_t^{\alpha}y_{n+1}(x,t)$ by

$$D_t^{\alpha}y_{n+1}(x,t)=F(x,t)-aD_{xx}^2\sum_{i=0}^{N}a_i\varphi_i(x)-b\sum_{i=0}^{N}a_i\varphi_i(x)-cG\sum_{i=0}^{N}a_i\varphi_i(x) \quad (21)$$

Then SCM convergences if the following are satisfied.

i. $(D_t^{\alpha}(y)-D_t^{\alpha}(Y), y-Y)\geq p\|y-Y\|^2, p>0, y, Y\epsilon H$
ii. For $\omega>0,\ \exists\, I(\omega)>0$ such that $\|y\|\leq\omega, \|Y\|\leq\omega, y, Y\in H$ then $((D_t^{\alpha}(y)-D_t^{\alpha}(Y), y-Y)\geq I(\omega)\|y-Y\|\|q\|, q\in H$

Proof

For $p>0,\ \ y, Y\in H$, we have

$$(D_t^\alpha(y) - D_t^\alpha(Y), y - Y) = \Big(\big(F(y,t) - aD_{yy}^2 \textstyle\sum_{i=0}^N a_i\varphi_i(y) - b\sum_{i=0}^N a_i\varphi_i(y) - cG\sum_{i=0}^N a_i\varphi_i(y)\big) - (F(x,t) - aD_{YY}^2 \sum_{i=0}^N a_i\varphi_i(Y) - b\sum_{i=0}^N a_i\varphi_i(Y) - cG\sum_{i=0}^N a_i\varphi_i(Y)), \quad y - Y\Big) \tag{22}$$

Applying the Schwartz Inequality, we get

$$\Big(\big(F(y,t) - aD_{yy}^2 \textstyle\sum_{i=0}^N a_i\varphi_i(y) - b\sum_{i=0}^N a_i\varphi_i(y) - cG\sum_{i=0}^N a_i\varphi_i(y)\big) - (F(x,t) - aD_{YY}^2 \sum_{i=0}^N a_i\varphi_i(Y) - b\sum_{i=0}^N a_i\varphi_i(Y) - cG\sum_{i=0}^N a_i\varphi_i(Y)), y - Y\Big)$$

$$\leq p\left\|\Big(\big(F(y,t) - aD_{yy}^2 \textstyle\sum_{i=0}^N a_i\varphi_i(y) - b\sum_{i=0}^N a_i\varphi_i(y) - cG\sum_{i=0}^N a_i\varphi_i(y)\big) - (F(x,t) - aD_{YY}^2 \sum_{i=0}^N a_i\varphi_i(Y) - b\sum_{i=0}^N a_i\varphi_i(Y) - cG\sum_{i=0}^N a_i\varphi_i(Y))\Big)\right\| \|y - Y\| \tag{23}$$

Using the mean value theorem, we obtain

$$\Big(\big(F(y,t) - aD_{yy}^2 \textstyle\sum_{i=0}^N a_i\varphi_i(y) - b\sum_{i=0}^N a_i\varphi_i(y) - cG\sum_{i=0}^N a_i\varphi_i(y)\big) - (F(x,t) - aD_{YY}^2 \sum_{i=0}^N a_i\varphi_i(Y) - b\sum_{i=0}^N a_i\varphi_i(Y) - cG\sum_{i=0}^N a_i\varphi_i(Y)), y - Y\Big) \geq \varepsilon\|y - Y\|^2 \tag{24}$$

where

$$\varepsilon = \frac{1}{2}p\omega^2$$

Hence

$$(D_t^\alpha(y) - D_t^\alpha(Y), y - Y) \geq p\|y - Y\|^2$$

Similarly, for $\omega > 0, \exists\, I(\omega) > 0$ such that $\|y\| \leq \omega$, $\|Y\| \leq \omega$, $y, Y \epsilon H$, then

$$(D_t^\alpha(y) - D_t^\alpha(Y), y - Y) = \Big(\big(F(y,t) - aD_{yy}^2 \textstyle\sum_{i=0}^N a_i\varphi_i(y) - b\sum_{i=0}^N a_i\varphi_i(y) - cG\sum_{i=0}^N a_i\varphi_i(y)\big)Y\Big)$$

$$\leq \omega^2\|y - Y\|\|q\|I(\omega)\|y - Y\|\|q\| \tag{25}$$

Thus, the second condition holds. This completes the proof.

Theorem 4.2

For a Banach space U, suppose the non-linear mapping $A: U \to U$ satisfy

$$\left\|A[y] - A[\bar{y}]\right\| \leq \beta\|y - \bar{y}\|, \qquad y, \bar{y} \epsilon\, U,$$

For some constant,$\beta < l$ there exist a unique fixed point such that the sequence

$$y_{n+1} = \alpha[y_n]$$

with arbitrary choice of $y_0 \epsilon y$, converges to the fixed point A_0,

$$\|y_i - \bar{y}_i\| \leq \|y_1 - \bar{y}_0\| \sum_{e-i=1}^{i-2} 2^e$$

Hence

$$A[y] = \left(F(y,t) - aD_{yy}^2 \sum_{i=0}^{N} a_i\varphi_i(y) - b\sum_{i=0}^{N} a_i\varphi_i(y) - cG\sum_{i=0}^{N} a_i\varphi_i(y)\right)$$

Proof

For each $y, \bar{y}\epsilon\, U$, then $\lim_{n\to\infty} \|A[y] - A[\bar{y}]\|$ exist, where A is non-linear mapping satisfying

$A: U \to U$.

Now for each $p\epsilon\, U$, where $p = (y, \bar{y})$, we have

$$\|A[y] - A[\bar{y}]\|^2 \leq A[y] - p, j(A[y] - p)$$

$$\geq \alpha_n\alpha < U_n - p, \qquad j(y_n - p) + (1 - \alpha_n) < \tau_n y_n - p,$$

$$\leq \alpha_n\|y_{n-1} - p\|\|y_n - p\| + (1 - \alpha_n)\|A[y] - A[\bar{y}]\|^2, n > 0$$

Simplifying, we have that

$$\|y_n - p\| \leq \|y_{n-1} - p\| \Rightarrow y_{n+1} = \beta[y_n]$$

Thus, the $\lim_{n\to\infty} \|A[y] - A[\bar{y}]\|$ exists, and so the sequence $\{y_n\}$ is bounded.

We next show that for some fixed constant the sequence $\{y_n\}$ converges to a definite fixed point

$$\|y_i - \bar{y}_i\| \leq \|y_1 - \bar{y}_0\| \sum_{e-i=1}^{i-2} 2^e$$

Thus, we have

$$\|y_n - \tau_n y_n\| = \alpha_n\|y_n - \tau_n y_n\| \to 0\,,\; n \to \infty \tag{26}$$

We now show that $\{y_n\}$ converges to some points in U.

In fact, it follows from (26) that there exist a subspace $\{y_r\}$ such that $\|y_r - \tau_r y_r\| \to 0$ as $n_r \to \infty$, $\tau_r y_r \to p$ and $y_r \to p$ (at some point y_0).

Consequently,

$$\|p - T_n\| \leq \|p - y_r\| + \|y_n - py_n\| + \|\tau_n y_n - \tau_n T_n\|$$
$$= \|y_n - py_n\| \leq \|y_{n+1} - y_n\| \sum_{j-i=1}^{j-2} 2^j \to 0$$

This implies that $T_p = p$. Since $y_n \to p$ and $\lim_{n\to\infty} \|A[y] - A[\bar{y}]\|$ exist, we have that $y_n \to p$

5. Numerical Perspectives

Consider the non-linear time fractional KGE (Amin *et al.*, 2020)

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + u^2(x,t) = f(x,t), \qquad 0 < t \leq 1, \ 0 < x \leq 1$$

with

$$f(x,t) = \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2} - \alpha\right)} (1-x)^{\frac{5}{2}} t^{\frac{3}{2}-\alpha} - \frac{15}{4}(1-x)^{\frac{1}{2}} t^{\frac{3}{2}} + (1-x)^5 t^3$$

the initial/end conditions can be extracted from the analytical/exact solution. The exact solution is given as:

$$u(x,t) = (1-x)^{\frac{5}{2}} t^{\frac{3}{2}-\alpha}$$

using the Mamadu-Njoseh polynomial with $\alpha = 1.4$ *and* 1.6 with $t = 0.7, 0.8$ *and* 0.9 and $T = 0.001$ with the adopted Spectral Collocation method in (15) and using the maple 18 Solver, the results are presented in the Tables below;

Table 4. 1: Maximum Error with $\alpha = 1.4$ and $t = 0.7$ and $T = 0.001$

$\boldsymbol{U(x,t)}$	**EXACT SOLUTION**	**MNSCM SOLUTION**	**ABSOLUTE ERROR (MNSCM)**	**ABSOLUTE ERROR (VIM)**
0.1	0.7415084	-0.0015411	7.4305×10^{-1}	4.0847×10^{-2}
0.2	0.5523760	-0.0015264	5.5390×10^{-1}	2.9602×10^{-2}
0.3	0.3955987	-0.0015111	3.9711×10^{-1}	2.9814×10^{-2}
0.4	0.2690840	-0.0014951	2.7058×10^{-1}	1.2515×10^{-1}
0.5	0.1705826	-0.0014783	1.7206×10^{-1}	2.4189×10^{-1}
0.6	0.0976472	-0.0014606	9.9108×10^{-2}	3.6375×10^{-1}
0.7	0.0475678	-0.0014421	4.9010×10^{-2}	4.7117×10^{-1}
0.8	0.0172617	-0.0014225	1.8684×10^{-2}	5.3613×10^{-1}
0.9	0.0030515	-0.0014018	4.4532×10^{-3}	5.0416×10^{-1}

Table 4.2: Maximum Error with $\alpha = 1.4$ and $t = 0.8$ and $T = 0.001$

$\boldsymbol{U(x,t)}$	**EXACT SOLUTION**	**MNSCM SOLUTION**	**ABSOLUTE ERROR (MNSCM)**	**ABSOLUTE ERROR (VIM)**
0.1	0.7514763	-0.0015411	7.5302×10^{-1}	4.0847×10^{-2}
0.2	0.5598014	-0.0015264	5.6133×10^{-1}	2.9602×10^{-2}
0.3	0.4009167	-0.0015111	4.0243×10^{-1}	2.9814×10^{-2}
0.4	0.2727012	-0.0014951	2.7420×10^{-1}	1.2515×10^{-1}
0.5	0.1728757	-0.0014783	1.7435×10^{-1}	2.4189×10^{-1}
0.6	0.0989598	-0.0014606	1.0042×10^{-1}	3.6375×10^{-1}
0.7	0.0482072	-0.0014421	4.9649×10^{-2}	4.7117×10^{-1}
0.8	0.0174938	-0.0014225	1.8916×10^{-2}	5.3613×10^{-1}
0.9	0.0030925	-0.0014018	4.4942×10^{-3}	5.0416×10^{-1}

Table 4.3: Maximum Error with $\alpha = 1.4$ and $t = 0.9$ and $T = 0.001$

$\boldsymbol{U(x,t)}$	**EXACT SOLUTION**	**MNSCM SOLUTION**	**ABSOLUTE ERROR (MNSCM)**	**ABSOLUTE ERROR (VIM)**
0.1	0.7603797	-0.0015411	7.6192×10^{-1}	4.0847×10^{-2}
0.2	0.5664339	-0.0015264	5.6796×10^{-1}	2.9602×10^{-2}
0.3	0.4056667	-0.0015111	4.0718×10^{-1}	2.9814×10^{-2}
0.4	0.2759322	-0.0014951	2.7743×10^{-1}	1.2515×10^{-1}
0.5	0.1749239	-0.0014783	1.7640×10^{-1}	2.4189×10^{-1}
0.6	0.1001323	-0.0014606	1.0159×10^{-1}	3.6375×10^{-1}
0.7	0.0487784	-0.0014421	5.0220×10^{-2}	4.7117×10^{-1}
0.8	0.0177011	-0.0014225	1.9124×10^{-2}	5.3613×10^{-1}
0.9	0.0031291	-0.0014018	4.5309×10^{-3}	5.0416×10^{-1}

Table 4.4: Maximum Error with $\alpha = 1.6$ and $t = 0.7$ and $T = 0.001$

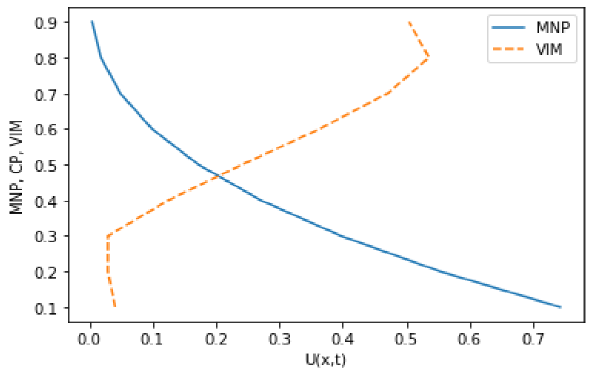
U(x,t)	EXACT SOLUTION	MNSCM SOLUTION	ABSOLUTE ERROR (MNSCM)	ABSOLUTE ERROR (VIM)
0.1	0.7963362	-0.0015411	7.9788×10^{-1}	9.5675×10^{-2}
0.2	0.5932192	-0.0015264	5.9475×10^{-1}	7.0445×10^{-2}
0.3	0.4248497	-0.0015111	4.2636×10^{-1}	5.6283×10^{-2}
0.4	0.2889804	-0.0014951	2.9048×10^{-1}	1.0526×10^{-1}
0.5	0.1831957	-0.0014783	1.8467×10^{-1}	2.2928×10^{-1}
0.6	0.1048673	-0.0014606	1.0633×10^{-1}	3.5653×10^{-1}
0.7	0.0510850	-0.0014421	5.2527×10^{-2}	4.6766×10^{-1}
0.8	0.0185381	-0.0014225	1.9961×10^{-2}	5.3485×10^{-1}
0.9	0.0032771	-0.0014018	4.6789×10^{-3}	5.0393×10^{-1}

Table 4.5: Maximum Error with $\alpha = 1.4$ and $t = 0.8$ and $T = 0.001$

$\boldsymbol{U(x,t)}$	**EXACT SOLUTION**	**MNSCM SOLUTION**	**ABSOLUTE ERROR (MNSCM)**	**ABSOLUTE ERROR (VIM)**
0.1	0.7857733	-0.0015411	7.8731×10^{-1}	7.5144×10^{-2}
0.2	0.5853505	-0.0015264	5.8688×10^{-1}	5.515×10^{-2}
0.3	0.4192143	-0.0015111	4.2073×10^{-1}	1.1516×10^{-2}
0.4	0.2851472	-0.0014951	2.8664×10^{-1}	2.3400×10^{-1}
0.5	0.1807657	-0.0014783	1.8224×10^{-1}	2.3400×10^{-1}
0.6	0.1034763	-0.0014606	1.0494×10^{-1}	3.5923×10^{-1}
0.7	0.0504074	-0.0014421	5.1849×10^{-2}	4.3533×10^{-1}
0.8	0.0182922	-0.0014225	1.9715×10^{-2}	5.3533×10^{-1}
0.9	0.0032336	-0.0014018	4.6354×10^{-3}	5.0402×10^{-1}

Table 4.6: Maximum Error with $\alpha = 1.4$ and $t = 0.9$ and $T = 0.001$

$\boldsymbol{U(x,t)}$	**EXACT SOLUTION**	**MNSCM SOLUTION**	**ABSOLUTE ERROR (MNSCM)**	**ABSOLUTE ERROR (VIM)**
0.1	0.7765725	-0.0015411	7.7811×10^{-1}	5.7040×10^{-2}
0.2	0.5784965	-0.0015264	5.8002×10^{-1}	4.1664×10^{-2}
0.3	0.4143056	-0.0015111	4.1582×10^{-1}	2.1175×10^{-2}
0.4	0.2818084	-0.0014951	2.8330×10^{-1}	1.1928×10^{-1}
0.5	0.1786491	-0.0014783	1.8013×10^{-1}	2.3816×10^{-1}
0.6	0.1022647	-0.0014606	1.0373×10^{-1}	3.6162×10^{-1}
0.7	0.0498172	-0.0014421	5.1259×10^{-2}	4.7014×10^{-1}
0.8	0.0180780	-0.0014225	1.9500×10^{-2}	5.3575×10^{-1}
0.9	0.0031958	-0.0014018	4.5975×10^{-3}	5.0409×10^{-1}



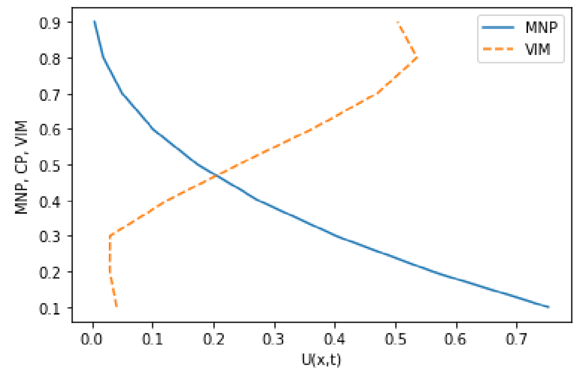
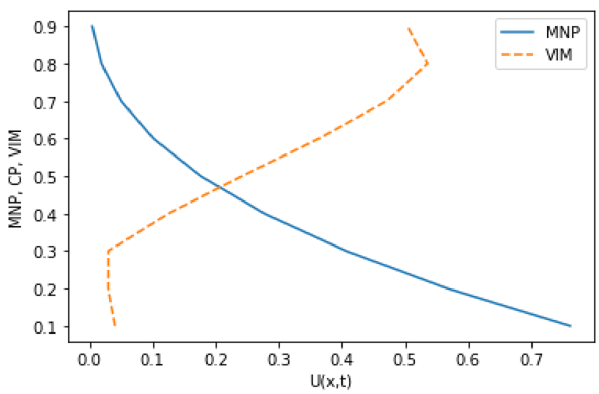


Figure 4.1: Maximum Error with α=1.4 and t=0.7 and T=0.001

Figure 4.2: Maximum Error with α=1.4 and t=0.8 and T=0.001



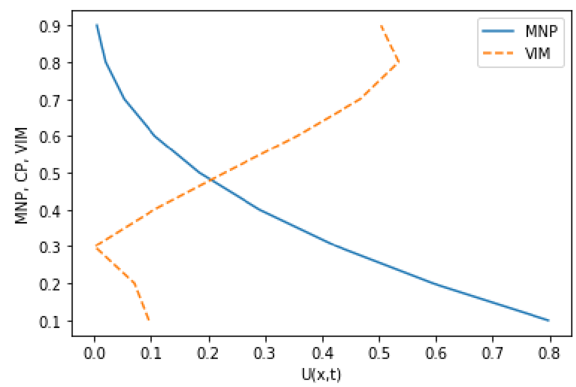
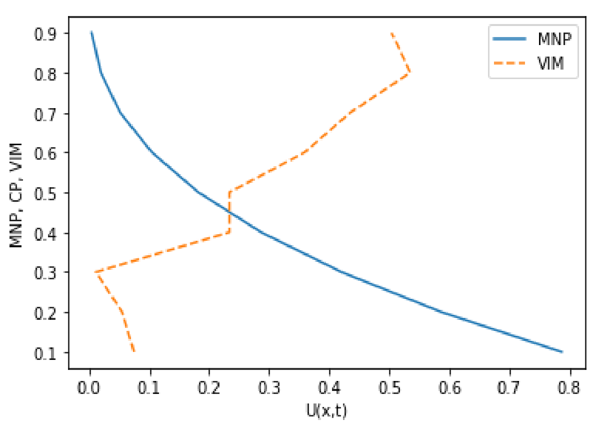


Figure 4.3: Maximum Error with α=1.4 and t=0.9 and T=0.001

Figure 4.4: Maximum Error with α=1.6 and t=0.7 and T=0.001



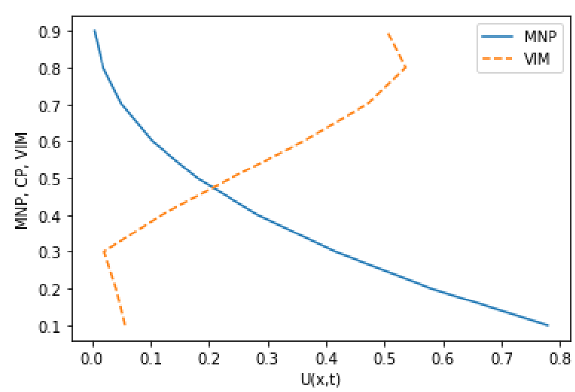


Figure 4.5: Maximum Error with α=1.6 and t=0.8 and T=0.001.

Figure 4.6: Maximum Error with α=1.6 and t=0.89 and T=0.001

6. Discussion of Results

We obtained some fascinating results in the cause of the implementation of the Spectral collocation method with the aid Mamadu-Njoseh polynomials as a trial function in the approximation of the actual solution of the fractional Klein-Gordon's equation. We have presented numerical evidence in tables and graphs with results compared with the Variational Iteration Method (VIM) as available in the literature.

It was observed that the rate of convergence of solutions is controlled by the parameter x and t for all cases considered as shown in Tables 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6. For emphasis, for $\mathrm{t} = 0.7, 0.8$ and 0.9 with $\alpha = 1.4\ and\ 1.6$, a maximum of error of order 10^{-3} was obtained for all Mamadu-Njoseh Spectral Collocation Method (MNSCM) as against the VIM with maximum error of order 10^{-2}. This suggests that as the value of x grows the maximum error with respect to MNSCM becomes smaller unlike the VIM where the maximum error becomes large. Thus, we can conclude that the rate of convergence of solutions of the spectral collocation via certain orthogonal (Mamadu-Njoseh polynomials) is controlled by the parameter t for optimal output. Also, the use of orthogonal polynomials gives better approximations than VIM.

7. Conclusion

It is very important to know that with use of numerical methods, there is hope for many mathematical models since many known analytical methods are difficult to solve these methods. Thus, our results have shown that the use of orthogonal polynomials such as Mamadu-Njoseh Polynomials as trial functions gives better approximations via spectral collocation method. On the basis of our analysis and computation, we strongly advocate the use of orthogonal polynomials as trial functions when executing the spectral collocation scheme for the solution of many nonlinear fractional partial differential equations.

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